Asymptotically optimal K_k-packings of dense graphs via fractional K_k-decompositions

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Abstract

Let H be a fixed graph. A *fractional H-decomposition* of a graph G is an assignment of nonnegative real weights to the copies of H in G such that for each $e \in E(G)$, the sum of the weights of copies of H containing e in precisely one. An *H-packing* of a graph G is a set of edge disjoint copies of H in G. The following results are proved. For every fixed $k > 2$, every graph with n vertices and minimum degree at least $n(1-1/9k^{10})+o(n)$ has a fractional K_k-decomposition and has a K_k-packing which covers all but $o(n^2)$ edges.

1 Introduction

All graphs considered here are finite, undirected and simple. For standard graph-theoretic terminology the reader is referred to [1]. Let H be a fixed graph. For a graph G, the *H-packing number*, denoted $\nu_H(G)$, is the maximum number of pairwise edge-disjoint copies of H in G. A function ψ from the set of copies of H in G to $[0,1]$ is a *fractional H-packing* of G if $\sum_{e \in H} \psi(H) \le 1$ for each $e \in E(G)$. For a fractional H-packing ψ, let $|\psi| = \sum_{H \in \binom{G}{H}} \psi(H)$. The *fractional H-packing number*, denoted $\nu^*_H(G)$, is defined to be the maximum value of $|\psi|$ over all fractional packings ψ. Notice that, trivially, $\nu^*_H(G) \ge \nu_H(G)$. In case $\nu_H(G) = e(G)/e(H)$ we say that G has an *H-decomposition*. In case $\nu^*_H(G) = e(G)/e(H)$ we say that G has a *fractional H-decomposition*. It is well known that computing $\nu_H(G)$ is NP-Hard for every fixed graph H with more than two edges in some connected component [3]. It is well known that computing $\nu^*_H(G)$ is solvable in polynomial time for every fixed graph H as this amounts to solving a (polynomial size) linear program.

The combinatorial aspects of the H-packing and H-decomposition problems have been studied extensively. Wilson in [11] has proved that whenever $n \ge n_0 = n_0(H)$, and K_n satisfies two obvious

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necessary divisibility requirements, K_n has an H-decomposition. The H-packing problem for K_n ($n \geq n(H)$) was solved [2], by giving a closed formula for $\nu(H, K_n)$. For graphs G which are not complete, giving sufficient conditions guaranteeing an H-decomposition or, at least, a packing covering all but a small fraction of the edges seems to be an extremely difficult task, even if we assume that G is dense. To be more precise, let us define the following problem and parameter.

Problem: Given a fixed graph H, determine c_H, which is the supremum of all possible constants c guaranteeing that every graph G with n vertices and minimum degree $\delta(G) \geq (1-c)(n-1)$ has $\nu_H(G) \geq (1 - o_n(1))e(G)/e(H)$.

Wilson's Theorem implies that $c_H \geq 0$ exists. In fact, $c_H \geq 0$ can also be derived from Rödl's result [10] which is weaker than Wilson's for graphs, but is more general since it applies to r-uniform hypergraphs as well. Gustavsson [5] has proved that $c_H > 0$ for every graph H. However, Gustavsson's lower bound for c_H is horribly close to 0. Already for $H = K_3$ it only gives $c_{K_3} > 10^{-24}$, and, more generally, if H has k vertices then $c_H > 10^{-37}k^{-94}$. Gustavsson's result does however, show that the minimum degree requirement, together with necessary divisibility conditions, guarantees an H-decomposition. The exact value of c_H is unknown for any fixed non-bipartite graph H. Notice that, trivially, $c_H = 1$ in case H is a bipartite graph as the Turán number of such graphs is $o(n^2)$. However, if we insist on having a packing of size $\lfloor e(G)/e(H) \rfloor$ then it is known that a minimum degree of $0.5n(1 + o_n(1))$ suffices for each fixed bipartite graph H (other than the trivial K_2) having a vertex of degree one (this includes all trees) and this is asymptotically tight [12].

In this paper we prove the first reasonable general lower bound for c_H.

Theorem 1.1 *For all $k \geq 3$, $c_{K_k} \geq 1/9k^{10}$.*

Although Theorem 1.1 is stated only for K_k, a simple argument given in the sequel shows that the same lower bound holds for any graph H with k vertices. Theorem 1.1 is deduced as a combination of two powerful theorems, the first of which is the following.

Theorem 1.2 *For all $k \geq 3$, any graph G with n vertices and $\delta(G) \geq n(1 - 1/9k^{10}) + o(n)$ has a fractional K_k-decomposition.*

The proof of Theorem 1.2 appears in the following section.

Recently, Haxell and Rödl [6] proved that the H-packing number and the fractional H-packing number are very close for dense graphs. A simpler and more general proof of their result appears in [13].

Theorem 1.3 *[Haxell and Rödl [6]] For any fixed graph H, if G has n vertices then $\nu_H^*(G) - \nu_H(G) = o(n^2)$.* ■

Theorem 1.2 gives that for sufficiently large n, any graph G with n vertices and $\delta(G) \geq n(1 - 1/9k^{10}) + o(n)$ has $\nu^*_{K_k}(G) = e(G)/e(K_k)$. Thus, by Theorem 1.3, it also has $\nu_{K_k}(G) \geq e(G)/e(K_k) - o(n^2) = (1 - o_n(1))e(G)/e(K_k)$. Consequently, $c_{K_k} \geq 1/9k^{10}$ and Theorem 1.1 follows.

Finally, we note that an $1/(k+1)$ upper bound for c_{K_k} is given in the final section together with some additional concluding remarks.

2 Proof of Theorem 1.2

Let $\mathcal{F}$ be a fixed family of graphs. An *$\mathcal{F}$-decomposition* of a graph G is a set L of subgraphs of G, each isomorphic to an element of $\mathcal{F}$, and such that each edge of G appears in precisely one element of L. Let K_t^- denote the complete graph with t vertices, missing one edge. Let $\mathcal{F}_k = \{K_k\ ,\ K_{2k-1}\ ,\ K^-_{2k-1}\}$. The proof of Theorem 1.2 is a corollary of the following stronger theorem.

Theorem 2.1 *For all $k \geq 3$, every graph with n vertices and minimum degree at least $n(1 - 1/9k^{10}) + o(n)$ has an $\mathcal{F}_k$-decomposition.*

The following simple lemma shows that Theorem 1.2 is a corollary of Theorem 2.1.

Lemma 2.2 *For all $k \geq 2$, the graphs K_{2k-1} and K^-_{2k-1} have a fractional K_k-decomposition.*

Proof: It is trivial that for all $k' \geq k$, $K_{k'}$ has a fractional K_k-decomposition. In Particular, K_{2k-1} has a fractional K_k-decomposition. Let $A = \{u, v\}$ denote the set of the two non-adjacent vertices of K^-_{2k-1}, and let B denote the set of the remaining $2k-3$ vertices. Each edge incident with A lies on $\binom{2k-4}{k-2}$ copies of K_k. Each edge with both endpoints in B lies on $2\binom{2k-5}{k-3}$ copies of K_k that contain a vertex of A. Since $\binom{2k-4}{k-2} = 2\binom{2k-5}{k-3}$, by assigning the value $1/\binom{2k-4}{k-2}$ to each copy of K_k containing a vertex of A, and assigning the value 0 to the remaining copies of K_k, we obtain a fractional K_k-decomposition of K^-_{2k-1}. ■

We now focus on proving Theorem 2.1. For the rest of this section, let $t = 2k-1$. Notice that the $o(n)$ term in the statement of Theorem 2.1 allows us to assume, whenever necessary, that n is sufficiently large. In the proof of Theorem 2.1 it will be convenient to use Wilson's Theorem [11] mentioned in the introduction. (We note that it is also possible to use Rödl's packing theorem [10] instead of Wilson's Theorem at the price of some complication in the proof). Wilson's Theorem applied to K_t states the following.

Lemma 2.3 *[Wilson] Let $t > 2$ be a positive integer. There exists $N = N(t)$ such that for all $n > N$ with $n \equiv 1, t \bmod t(t-1)$, there is a decomposition of K_n into K_t.*

We shall prove Theorem 2.1 under the relaxed assumption that $n \equiv 1 \bmod t(t-1)$. We first need to justify this relaxation. Indeed, if $1 < b \leq t(t-1)$ and $n \equiv b \bmod t(t-1)$ then we can perform

the following preprocessing. Let v be any vertex of G, and let $N(v)$ denote its neighborhood. Let $G[N(v)]$ be the subgraph induced by this neighborhood. Notice that $G[N(v)]$ has less than n vertices, but has minimum degree at least $n(1-2/9k^{10})+o(n)$. By the theorem of Hajnal and Szemerédi [7], such a high minimum degree for a graph with at most n vertices is far more than what is needed in order to guarantee that $G[N(v)]$ has a spanning subgraph G' with each component of G' being either a K_{k-1} or a K_{3k-4}. In fact, a minimum degree of at least $n(1-1/(3k-4))$ already guarantees the existence of such a G'. Now, if H is a K_{k-1} component of G' then $H \cup \{v\}$ is a K_k copy of G. If H is a K_{3k-4} component of G' then $H \cup v$ contains two edge-disjoint subgraphs, one being K_k and the other being K_t^-, and with all $3k-4$ edges between v and the vertices of H absorbed. We thus have a set of edge-disjoint subgraphs of G, each being either K_k or K_t^-, and that absorb all edges incident with v. Deleting v and the edges of these subgraphs we remain with a graph with $n-1$ vertices and minimum degree at least $\delta(G)-(3k-4) \ge (n-1)(1-1/9k^{10})+o(n-1)$. Repeating this process at most $b-1$ times we eventually have a graph with $n' = n-b+1 \equiv 1 \bmod t(t-1)$ vertices and minimum degree at least $\delta(G)-(3k-4)((2k-1)(2k-2)-1) \ge n'(1-1/9k^{10})+o(n')$, which satisfied our relaxed assumption. Our preprocessing shows that any $\mathcal{F}_k$-decomposition of this resulting n'-vertex graph can be extended to an $\mathcal{F}_k$-decomposition of the original n-vertex graph.

Proof of Theorem 2.1 We may assume that $n \equiv 1 \bmod t(t-1)$ and that, whenever necessary, n is sufficiently large as a function of k (and hence t). In particular, $n > N(t)$ where $N(t)$ is the constant from Lemma 2.3. Fix a K_t-decomposition of K_n. Namely, let D be a family of t-sets of $[n] = \{1, \ldots, n\}$ such that each pair appears in precisely one element of D. Such a D is also called a t-design. Notice that $|D| = \binom{n}{2}/\binom{t}{2}$. For a permutation π of $[n]$, and for $S \in D$, let $S_\pi = \{\pi(j) \ : \ j \in S\}$. Hence, $D_\pi = \{S_\pi \ : \ S \in D\}$ is also a t-design. Let G be an n-vertex graph with vertex set $[n]$. For a permutation π of $[n]$ let G_π be the family of $\binom{n}{2}/\binom{t}{2}$ edge-disjoint subgraphs of G whose elements are the induced subgraphs of G on S_π, for all $S \in D$. Notice that if $G = K_n$ then, trivially, G_π is a K_t-decomposition for each π, but if $G \neq K_n$, G_π contains elements that are not isomorphic to K_t. The following is a simple corollary of Lemma 2.3.

Corollary 2.4 *Let $0 < \alpha < 1$ be fixed. Let G be a graph with $n > N(t)$ vertices, $n \equiv 1 \bmod t(t-1)$. If $\delta(G) \ge (1-\alpha)n$ and π is any permutation of $[n]$ then G_π has at least $(1-o(1))n^2(\frac{1}{t(t-1)} - \frac{\alpha}{2})$ elements isomorphic to K_t and at most $(1+o(1))n^2(\frac{\alpha}{2})(\binom{t}{2}-1)$ edges appear in elements of G_π that are not isomorphic to K_t.*

Proof: The number of non-edges of G is at most $\binom{n}{2} - (1-\alpha)n^2/2$. Thus, G_π has at most $\binom{n}{2} - (1-\alpha)n^2/2$ elements that are not K_t and therefore G_π has at least $\binom{n}{2}/\binom{t}{2} - \binom{n}{2} + (1-\alpha)n^2/2$ elements isomorphic to K_t and at most $(\binom{n}{2} - (1-\alpha)n^2/2)(\binom{t}{2}-1)$ edges of G are in non-K_t elements of G_π. ■

Assume that $\delta(G) \ge n(1-1/9k^{10})+o(n)$. Our goal is to show that there exists a permutation π, such that G_π has some "nice" properties. Let A_π denote the set of edges of G that appear in

non-K_t elements of G_π. By Corollary 2.4, with $\alpha = 1/9k^{10}$,

$$|A_\pi| \le (1+o(1))n^2 \left(\frac{1}{18k^{10}}\right)\left(\binom{t}{2}-1\right) \le (1+o(1))\frac{n^2}{9k^8}. \tag{1}$$

Consider the spanning subgraph of G consisting of the edges of A_π. It will not be confusing to denote this subgraph by A_π as well. Let $F_\pi \subset G_\pi$ be the set of K_t-elements of G_π. Put $r = \binom{k}{2} - 1$. We say that an r-subset $S = \{H_1, \ldots, H_r\}$ of F_π is *good for* $e \in A_\pi$ if we can select edges $f_i \in H_i$ such that $\{f_1, \ldots, f_r, e\}$ is the set of edges of a K_k in G. We say that π is *good* if for each $e \in A_\pi$ there exists an r-subset $S(e)$ of F_π such that $S(e)$ is good for e and such that if $e \neq e'$ then $S(e) \cap S(e') = \emptyset$.

Lemma 2.5 *If π is good then G has an $\mathcal{F}_k$-decomposition.*

Proof: For each $e \in A_\pi$, pick a copy of K_k in G containing e and precisely one edge from each element of $S(e)$. As each element of $S(e)$ is a K_t, deleting one edge from such an element results in a K_t^-. We therefore have $|A_\pi|$ copies of K_k and $|A_\pi|(\binom{k}{2} - 1)$ copies of K_t^-, all being edge disjoint. The remaining element of F_π not belonging to any of the $S(e)$ are each a K_t, and they are edge-disjoint from each other and from the previously selected K_k and K_t^-. ■

Our goal in the remainder of this section is, therefore, to show that there exists a good π. We use probabilistic and counting arguments to derive this fact. We will show that with positive probability, a randomly selected π is good. We begin by showing that with high probability, a randomly selected π has the property that A_π has a relatively small maximum degree.

Let v be any vertex of G, and let $E(v)$ denote the set of edges incident with v. By our assumption, $|E(v)| \ge (1 - 1/9k^{10})n$. Let $E_\pi(v) = E(v) \cap A_\pi$. Notice that if π is selected uniformly at random then $|E_\pi(v)|$, which is the degree of v in A_π, is a random variable. We say that a subset $S \subset E(v)$ is *separated by* π if each edge of S belongs to a different element of G_π. Let $\beta = 3/k^8$ and consider any fixed set $S \subset E(v)$ with $|S| = \lfloor \beta n \rfloor$. We shall prove that the probability that $S \subset E_\pi(v)$ *and* that S is separated by π is much smaller than the total number of subsets of $E(v)$ with size $\lfloor \beta n \rfloor$. Thus, the probability that the degree of v in A_π exceeds $t\beta n$ is also very small (in fact, much smaller than $1/n$), and consequently, the maximum degree of A_π is at most $t\beta n$ almost surely. Let $S = \{e_1, \ldots, e_m\}$ where $m = \lfloor \beta n \rfloor$ and let $e_i = (v, v_i)$. Let $S_\pi(i)$ denote the element of G_π to which the edge e_i belongs, $i = 1, \ldots, m$. Notice that S is separated by π if and only if $S_\pi(i) \neq S_\pi(j)$ for all $i \neq j$. Clearly, by using conditional probabilities we have

$$\Pr[(S \subset E_\pi(v)) \wedge (S \text{ is separated by } \pi)] = \tag{2}$$

$$\prod_{i=1}^{m} \Pr[(e_i \in A_\pi) \wedge (\forall j < i,\ S_\pi(i) \neq S_\pi(j)) \mid (\{e_1, \ldots, e_{i-1}\} \subset A_\pi) \wedge (S_\pi(j) \neq S_\pi(j'),\ 1 \le j < j' < i)].$$

We shall prove that each term in the product appearing in the r.h.s. of the last equation is small. For this purpose we require a lemma which quantifies the fact that in a graph with high minimum degree every edge appears on many copies of K_t.

Lemma 2.6 *If G^* is a graph with n^* vertices and minimum degree at least $n^* - r$ then every edge of G^* appears on at least $\frac{1}{(t-2)!}\prod_{i=2}^{t-1}(n^* - ir)$ distinct copies of K_t.*

We prove the lemma by induction on t. For $t = 2$ and $t = 3$ the lemma is obvious. Assume the lemma holds for all $t' < t$. Let $e = (u, v)$ be an edge of G^*. Let $N(u, v)$ denote the set of common neighbors of u and v. Clearly, $|N(u, v)| \geq n^* - 2r$. Let $G^{**} = G^*[N(u, v)]$. The minimum degree of G^{**} is at least $|N(u, v)| - r$. It follows that G^{**} has at least $(n^* - 2r)(n^* - 3r)/2$ edges. The number of distinct copies of K_{t-2} in G^{**} is equal to the number of distinct copies of K_t containing e in G^*. By the induction hypothesis, each edge of G^{**} appears in at least $\frac{1}{(t-4)!}\prod_{i=4}^{t-1}(n^* - ir)$ distinct copies of K_{t-2}. Since each copy of K_{t-2} is counted $(t-2)(t-3)/2$ times the number of distinct copies of K_{t-2} in G^{**} is at least

$$\frac{(n^* - 2r)(n^* - 3r)}{2}\frac{1}{(t-4)!}\prod_{i=4}^{t-1}(n^* - ir)\frac{2}{(t-2)(t-3)} = \frac{1}{(t-2)!}\prod_{i=2}^{t-1}(n^* - ir)$$

as required. ■

Corollary 2.7 *If G^* is a graph with n^* vertices and minimum degree at least $n^*(1 - \gamma)$ then, for every edge e of G^*, the probability that a randomly selected t-vertex subgraph of G^* that contains e is not a K_t is at most $1 - (1 - t\gamma)^{t-2}$.*

Proof: There are precisely $\binom{n^*-2}{t-2}$ subgraphs with t vertices that contain the edge e. By Lemma 2.6, with $r = \gamma n^*$, the number of K_t-subgraphs that contain e is at least

$$\frac{1}{(t-2)!}\prod_{i=2}^{t-1}(n^* - i\gamma n^*) > \frac{1}{(t-2)!}(n^*)^{t-2}(1 - t\gamma)^{t-2} > \binom{n^* - 2}{t - 2}(1 - t\gamma)^{t-2}.$$

Thus, the probability that a randomly selected t-vertex subgraph of G^* that contains e is not a K_t is at most $1 - (1 - t\gamma)^{t-2}$. ■

Corollary 2.7 enables us to estimate the terms in the r.h.s. of (2). Let $Y_\pi(j)$ be the set of t vertices of the element $S_\pi(j)$. Notice that given the knowledge that $S_\pi(j) \neq S_\pi(j')$ for $1 \leq j < j' < i$ implies, in particular, the knowledge that $Y_\pi(j) \cap Y_\pi(j') = \{v\}$. Thus, if $W = \cup_{j=1}^{i-1} Y_\pi(j)$ then $|W| = (i - 1)(t - 1) + 1$. Thus, we know the size of W. In order to prove an upper bound on each term of the r.h.s. of (2) it suffices to prove an upper bound on $\Pr[(e_i \in A_\pi) \wedge (S_\pi(i) \cap W = \{v\})|W]$ whose value does not depend on the specific set W, but which may, and will, depend on the prior

knowledge that $|W| = (i-1)(t-1)+1$. Indeed, let G^* be the induced subgraph of G obtained by deleting the set of vertices $W - \{v\}$. Notice that G^* has $n^* = n - (i-1)(t-1)$ vertices. Clearly, $\Pr[(e_i \in A_\pi) \wedge (S_\pi(i) \cap W = \{v\}) | W]$ is precisely the probability that a randomly selected t-vertex subgraph of G^* containing e_i is not a K_t. Using the fact that $(i-1)(t-1) < mt << n/2$, we have that the minimum degree of G^* is a least $n^* - n/9k^{10} \geq n^*(1 - 2/9k^{10})$. Using $\gamma = 2/9k^{10}$, we have by Corollary 2.7 that $\Pr[(e_i \in A_\pi) \wedge (S_\pi(i) \cap W = \{v\}) | W] \leq 1 - (1-t\gamma)^{t-2}$. Consequently each term in (2) is bounded from above by $1 - (1-t\gamma)^{t-2}$. We therefore have

$$\Pr[(S \subset E_\pi(v)) \wedge (S \text{ is separated by } \pi)] \leq (1 - (1-t\gamma)^{t-2})^m. \tag{3}$$

Lemma 2.8 *With probability $1 - o(1)$, A_π has maximum degree at most $6n/k^7$.*

Proof: Since the maximum degree of each element of G_π is at most $t-1$ we have, by the definition of $E_\pi(v)$, that there is $S \subset E_\pi(v)$ such that S is separated by π and $|S| \geq |E_\pi(v)|/(t-1)$. Thus, it suffices to show that for each vertex v, $E_\pi(v)$ has no subset separated by π of size greater than $3n/k^8$ with probability $1 - o(1/n)$ since this implies that $|E_\pi(v)| \leq (t-1)3n/k^8 \leq 6n/k^7$ with probability $1 - o(1/n)$ and hence the maximum degree of A_π is at most $6n/k^7$ with probability $1 - o(1)$. Indeed, by (3) the probability that $E_\pi(v)$ has a subset separated by π of size $m = \lfloor \beta n \rfloor = \lfloor 3n/k^8 \rfloor$ is at most $\binom{n-1}{m}(1 - (1-t\gamma)^{t-2})^m$ where $\gamma = 2/9k^{10}$. We therefore have

$$\binom{n-1}{m}(1-(1-t\gamma)^{t-2})^m \leq \left(\frac{1}{\beta^\beta(1-\beta)^{1-\beta}}\left(1-(1-\frac{4}{9k^9})^{2k-3}\right)^\beta\right)^n$$

$$\leq \left(\frac{1}{\beta^\beta(1-\beta)^{1-\beta}}\left(1-(1-\frac{1}{k^8})\right)^\beta\right)^n = \left(\frac{1}{3^\beta(1-\beta)^{1-\beta}}\right)^n = o\left(\frac{1}{n}\right).$$

■

By Lemma 2.8, we may *fix* a permutation π for which A_π has maximum degree at most $6n/k^7$. Let $A_\pi = \{e_1, \ldots, e_m\}$. We perform the following algorithm which has m iterations. In the i'th iteration we pick an r-subset $S(e_i)$ of F_π which is good for e_i and which satisfies the following two properties:

1. $S(e_i) \cap S(e_j) = \emptyset$ for all $j = 1, \ldots, i-1$.

2. For each v, let $f_i(v)$ denote the number of edges incident with v and which belong to some element of $S(e_j)$, $j \leq i$, and where v is not an endpoint of e_j. Then, $f_i(v) \leq n/(2k)$.

Notice that if we can complete all m iterations of the algorithm then the first requirement guarantees that π is good and hence, by Lemma 2.5 we are done. The second requirement is needed in order to guarantee that the algorithm will, indeed, complete all m iterations. We therefore need to prove the following lemma.

Lemma 2.9 *If A_π has maximum degree at most $6n/k^7$ then the algorithm completes all m iterations.*

Proof: The most difficult case is to prove that the m'th iteration can also be completed, assuming all previous iterations have completed. Let $e_m = (u, v)$. We define several parameters. Let a_1 denote the number of K_k copies of G that contain e_m. Let a_2 denote the number of K_k copies of G that contain e_m and also contain two edges from the same element of F_π. Let a_3 denote the number of K_k copies of G that contain e_m and also contain another edge from A_π. Let E_m denote the set of $(m-1)r\binom{t}{2}$ edges in all elements of $\cup_{i=1}^{m-1} S(e_i)$. Let a_4 denote the number of K_k copies of G that contain e_m and also contain an edge of E_m. Let V_m be the subset of vertices of G where $x \in V_m$ if and only if $x \neq u, v$ and $f_{m-1}(x) > n/k - r(t-1)$. Let F_m be the set of all edges of all copies of F_π that contain at least one vertex of V_m. Let a_5 denote the number of K_k copies of G that contain e_m and also contain an edge of F_m.

We claim that if $a_1 > a_2 + a_3 + a_4 + a_5$ then the m'th iteration can be completed. Indeed, if this is the case then by the definitions of a_2 and a_3 there exists a copy of K_k in G which contain e_m, and whose other r edges all belong to distinct elements of F_π, say, $S(e_m) = \{H_1, \ldots, H_r\}$. Furthermore, by the definition of a_4 we may also assume that no H_i is an element of a previous $S(e_j)$ for $j < m$, and hence $S(e_m) \cap S(e_j) = \emptyset$ for all $j = 1, \ldots, m-1$. Finally, by the definition of a_5 we may assume that no H_i contains a vertex of V_m. Thus, for $x \in V_m$ we have $f_m(x) = f_{m-1}(x) \leq n/(2k)$ by our assumption. By definition, since u, v are incident with e_m we have $f_m(v) = f_{m-1}(v) \leq n/(2k)$ and $f_m(u) = f_{m-1}(u) \leq n/(2k)$. For $x \notin V_m \cup \{u, v\}$ notice that we have $f_m(x) \leq f_{m-1}(x) + r(t-1) \leq n/(2k)$ as well.

It remains to show that $a_1 > a_2 + a_3 + a_4 + a_5$. We now estimate these parameters. A similar proof to that of Corollary 2.7 (where we use n instead of n^*, k instead of t and $\gamma = 1/9k^{10}$ immediately gives

$$a_1 \geq \binom{n-2}{k-2}(1 - \frac{1}{9k^9})^{k-2} \geq n^{k-2}\frac{0.95}{(k-2)!}(1 - o(1)). \tag{4}$$

Consider a pair of edges f_1, f_2 that belong to the same element of F_π. If they are both independent from e_m then $\{e_m, f_1, f_2\}$ spans at least five vertices. Thus, there are at most $\binom{n-5}{k-5}$ copies of K_k that contain all three of them. As there are less than $|F_\pi|t^4 = \Theta(n^2)$ possible choices for pairs f_1, f_2 the overall number of such copies is $O(n^{k-3})$. If f_1, f_2 are not independent from e_m then we must have that $\{e_m, f_1, f_2\}$ spans at least four vertices. Thus, there are at most $\binom{n-4}{k-4}$ copies of K_k that contain all three of them. However, there are less than $2n$ edges not independent from e_m, so the overall number of choices for f_1, f_2 is only $O(n)$. Overall there are, again, only $O(n^{k-3})$ such copies. We have proved that

$$a_2 = O(n^{k-3}). \tag{5}$$

Consider an edge $f \in A_\pi$ with $f \neq e_m$. If f and e_m are independent then there are at most $\binom{n-4}{k-4}$ copies of K_k containing both of them. Overall, there are at most $m\binom{n-4}{k-4}$ such copies. If f and e_m

are not independent then there are at most $\binom{n-3}{k-3}$ copies of K_k containing both of them. However, the maximum degree of A_π is at most $6n/k^7$ and hence there are at most $12n/k^7$ choices for f. Using (1) we therefore have

$$a_3 \le m\binom{n-4}{k-4} + \frac{12n}{k^7}\binom{n-3}{k-3} \le (1+o(1))\frac{n^2}{9k^8}\binom{n-4}{k-4} + \frac{12n}{k^7}\binom{n-3}{k-3} \tag{6}$$
$$\le n^{k-2}\left(\frac{1}{9k^8(k-4)!} + \frac{12}{k^7(k-3)!}\right)(1+o(1)).$$

Notice that $|E_m| = (m-1)r\binom{t}{2} \le mk^4 \le (1+o(1))\frac{n^2}{9k^8}k^4 \le (1+o(1))\frac{n^2}{9k^4}$. If $f \in E_m$ is independent from e_m then they appear together in at most $\binom{n-4}{k-4}$ copies of K_k. If f and e_m are not independent, then they may appear together in at most $\binom{n-3}{k-3}$ copies of K_k. Suppose, w.l.o.g., that $f = (u,x)$. Since $f_{m-1}(u) \le n/(2k)$ we know that there are at most $n/(2k) + qr(t-1) \le n/(2k) + 2k^4q$ choices for f where q is the number of edges e_j with $j < i$ and which have u as an endpoint. However, $q \le \Delta(A_\pi) \le 6n/k^7$. Thus, we have that

$$a_4 \le (1+o(1))\frac{n^2}{9k^4}\binom{n-4}{k-4} + \left(\frac{n}{2k} + 2k^4\frac{6n}{k^7}\right)\binom{n-3}{k-3} \tag{7}$$
$$\le n^{k-2}\left(\frac{1}{9k^4(k-4)!} + \frac{1}{2k(k-3)!} + \frac{12}{k^3(k-3)!}\right)(1+o(1)).$$

In order to estimate a_5 we need to estimate the size of V_m. Since $|E_m| \le (1+o(1))\frac{n^2}{9k^4}$ we have that $|V_m|(n/(2k) - r(t-1)) < (1+o(1))2n^2/9k^4$. Thus, $|V_m| \le (1+o(1))4n/9k^3$. Trivially, each vertex appears it at most $(n-1)/(t-1)$ elements of F_π. Thus, the overall number of elements of F_π containing an element of V_m is at most $(1+o(1))4n^2/(9k^3(t-1))$. It follows that $|F_m| \le (1+o(1))\binom{t}{2}4n^2/(9k^3(t-1))$. As before, for $f \in F_m$ which is independent of e_m there are at most $\binom{n-4}{k-4}$ copies of K_k containing both f and e_m. If $f \in F_m$ is not independent with e_m then assume $f = (u,x)$. We therefore must have some $y \in V_m$ (possibly $y = x$) such that (u,y) and (u,x) are in the same element of F_π. It follows that there are at most $2|V_m|(t-1)$ choices for f which shares an endpoint with e_m and, as before, each such f appears together with e_m in at most $\binom{n-3}{k-3}$ copies of K_k. We therefore have

$$a_5 \le (1+o(1))\binom{t}{2}\frac{4n^2}{9k^3(t-1)}\binom{n-4}{k-4} + 2(t-1)(1+o(1))\frac{4n}{9k^3}\binom{n-3}{k-3} \tag{8}$$
$$\le n^{k-2}\left(\frac{4}{9k^2(k-4)!} + \frac{16}{9k^2(k-3)!}\right)(1+o(1)).$$

By inequalities (4), (5),(6),(7) and (8) we have that $a_1 > a_2 + a_3 + a_4 + a_5$ since

$$0.95 > \frac{(k-2)(k-3)}{9k^8} + \frac{12(k-2)}{k^7} + \frac{(k-2)(k-3)}{9k^4} + \frac{k-2}{2k} + \frac{12(k-2)}{k^3} + \frac{4(k-2)(k-3)}{9k^2} + \frac{16(k-2)}{9k^2}$$

holds for all $k \ge 3$. ■

We have now completed the proof of Theorem 2.1. ■

3 Concluding remarks and open problems

- Although Theorem 1.1 and Theorem 1.2 are stated only for K_k, it is easy to see that these theorems also hold for any k-vertex graph. Indeed, if H has k vertices then, trivially, K_k has a fractional H-decomposition. Thus, any graph which has a fractional K_k-decomposition also has a fractional H-decomposition. It follows that Theorem 1.2 holds also for H. Combining Theorem 1.3 and Theorem 1.2 as explained in the introduction gives that Theorem 1.1 also holds for H.

- For all $k \geq 3$, Theorem 1.1 gives a lower bound of $1/9k^{10}$ for c_{K_k}. We now prove that $c_{K_k} \leq 1/(k+1)$. We will prove something slightly stronger; For all $\epsilon > 0$ there exists $\delta > 0$ and a graph G with n vertices and $\delta(G) \geq n(1-1/(k+1)) - \epsilon n$ for which $\nu^*_{K_k}(G) \leq (1-\delta)(e(G)/e(K_k))$. We will use a modification of a construction from [5] for this purpose. Let s be a positive integer and let H_s be any r-regular graph with $2s(k^3-k)$ vertices and $r = 4s(k^2-k) - d$ where $d = \lfloor \epsilon 2s(k^3-k)(k-1) \rfloor$. Such graphs clearly exist for s sufficiently large as a function of k and ϵ. Let G_s be the graph constructed by blowing up each vertex of K_{k-1} to a copy of H_s. Clearly, G_s has $n = 2s(k^3-k)(k-1)$ vertices. G_s is regular of degree $\delta = r + 2s(k^3-k)(k-2)$. Notice that $\delta = n(1-1/(k+1)) - \lfloor \epsilon n \rfloor$. However, any K_k in G must contain an edge from one of the blown up copies of H_s. It follows that

$$\nu^*_{K_k}(G) \leq (k-1)e(H_s) = (k-1)s(k^3-k)(4s(k^2-k)-d).$$

  But

$$e(G) = (2s(k^3-k))^2 \binom{k-1}{2} + (k-1)s(k^3-k)(4s(k^2-k)-d).$$

  It follows that $\nu^*_{K_k}(G) \leq (1-\delta)(e(G)/\binom{k}{2})$ where $\delta = \delta(\epsilon, k)$.

- Theorem 1.2 can be implemented in polynomial time, since given an input graph G with $\delta(G) \geq (1-1/9k^{10})n + o(n)$ we are guaranteed by Lemma 1.2 that the solution to the linear program computing $\nu^*_{K_k}(G)$ is $e(G)/e(K_k)$. It is shown in [6] that Theorem 1.3 can be implemented in polynomial time, namely, any fractional packing can be converted in polynomial time to an integral packing whose value differs from $\nu^*_{K_k}(G)$ by $o(n^2)$. It follows that Theorem 1.1 can also be implemented in polynomial time. The same arguments hold if we replace K_k with any k-vertex graph H.

- It is plausible Theorem 1.1 can also be proved for hypergraphs. Namely, for any positive integers k and r with $k > r$ there exists $\delta = \delta(k,r)$ such that the following holds. Let K_k^r denote the complete r-uniform hypergraph on k vertices. Then, any n-vertex r-uniform hypergraph H with minimum degree at least $\binom{n-1}{r-1}(1-\delta(k,r))$ has a K_k^r-packing of size at least $(1-o(1))e(H)/e(K_k^r)$. We already have some partial results in this direction. The proof

in the hypergraph case turns out to be significantly more involved than in the graph-theoretic case. Details will appear in a separate paper.

- The constant $1/9k^{10}$ is chosen to accommodate all $k \geq 3$ throughout all the lemmas. By carefully reviewing all computations for the case $k = 3$ it is easy to get $c_{K_3} \geq 1/90000$ which is 6 times better than the general constant. We do not bother with the details since there is no indication that this improved lower bound is close to the truth. In fact, it the conjecture of Nash-Williams [9] is true then $c_{K_3} = 1/4$.

- Theorem 1.1 should be compared to other packing results which guarantee a tight packing. If all edges of the graph G lie on $\alpha n^{k-2}(1 + o(1))$ copies of K_k for some α then the result of Frankl and Rödl [4] guarantees a packing of size $(1 - o(1))e(G)/e(K_k)$. However, for any $\delta > 0$, there are graphs with minimum degree at least $(1-\delta)n$ for which no such α exists. For any $\delta > 0$, applying the result of Kahn [8] to graphs with minimum degree at least $(1 - \delta)n$ yields a packing of size $(1 - \beta)e(G)/e(K_k)$, where β is a *constant* that depends on δ and k and tends to zero as δ tends to zero.

- Theorem 2.1 gives a nontrivial minimum degree requirement which guarantees the existence of an $\mathcal{F}$-decomposition for the family $\mathcal{F} = \{K_k, K_{2k-1}, K^-_{2k-1}\}$. It is interesting to find other more general families $\mathcal{F}$ for which nontrivial minimum degree conditions guarantee an $\mathcal{F}$-decomposition, and which do not rely on the horrible bounds from [5]. Notice that if $\mathcal{F}$ contains a bipartite graph with minimum degree one this problem is solved in [12].

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